Diploma Programme
Programme du diplôme
Programa del Diploma

Markscheme

May 2017

Further mathematics

Higher level

Paper 2

16 pages

International Baccalaureate®
Baccalauréat International
Bachillerato Internacional

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a valid **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to RM™ Assessor instructions and the document "**Mathematics HL: Guidance for e-marking May 2017**". It is essential that you read this document before you start marking. In particular, please note the following:

- Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.
- If a part is **completely correct**, (and gains all the "must be seen" marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.
- All the marks will be added and recorded by RM™ Assessor.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *eg* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*eg* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.*, do **not** split the marks.
- Once a correct answer to a question or part-question is seen, ignore further working. However, if further working indicates a lack of mathematical understanding do not award the final ***A1***. An exception to this may be in numerical answers, where a correct exact value is followed by an incorrect decimal. However, if the incorrect decimal is carried through to a subsequent part, and correct ***FT*** working shown, award ***FT*** marks as appropriate but do not award the final ***A1*** in that part.

Examples

	Correct answer seen	Further working seen	Action
1.	$8\sqrt{2}$	$5.65685...$ *(incorrect decimal value)*	Award the final ***A1*** *(ignore the further working)*
2.	$\frac{1}{4}\sin 4x$	$\sin x$	Do not award the final ***A1***
3.	$\log a - \log b$	$\log(a-b)$	Do not award the final ***A1***

3 ***N*** **marks**

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 **Implied marks**

*Implied marks appear in **brackets eg (M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 **Follow through marks**

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 **Misread**

*If a candidate incorrectly copies information from the question, this is a misread (**MR**).*
*A candidate should be penalized only once for a particular misread. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses one mark.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

*Unless the question specifies otherwise, **accept** equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

$f'(x) = \left(2\cos(5x-3)\right)5 \ \left(=10\cos(5x-3)\right)$ ***A1***

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

*Candidates should **NO LONGER** be penalized for an accuracy error **(AP)**.*

*If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for **FT**.*

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

A GDC is required, but calculators with symbolic manipulation features (for example, TI-89) are not allowed.

Calculator notation
The Mathematics HL guide says:
Students must always use correct mathematical notation, not calculator notation.
Do **not** accept final answers written using calculator notation. However, do not penalize the use of calculator notation in the working.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

1. (a) (i) there is an Eulerian trail because G contains exactly two vertices of odd order ***A1***
there is no Eulerian circuit because G contains vertices of odd order ***A1***

(ii) the trail must start at B and end at E (or vice versa) ***(R1)***
BAFBCFECDE ***R1***

[4 marks]

(b)

Step	Vertices labelled	Working values	
1	A	[A0], B-4, F-3	***M1***
2	A, F	[A0], [F3], B-4, C-10, E-15	***A1***
3	A, F, B	[A0], [F3], [B4], C-7, E-15	***A1***
4	A, F, B, C	[A0], [F3], [B4], [C7], D-14, E-10	***A1***
5	A, F, B, C, E	[A0], [F3], [B4], [C7], [E10], D-12	***A1***
6	A, F, B, C, E, D	[A0], [F3], [B4], [C7], [E10], [D12]	***(A1)***

the path of minimum total weight is ABCED with total weight 12 ***A1N2***

Note: Award full marks if the correct path is given with correct total weight if an annotated graph is given that represents the Dijkstra algorithm.

[7 marks]

Total [11 marks]

2. (a) Reflexive: xRx ***(M1)***
because $x^{-1}x = e \in H$ ***R1***
therefore reflexive ***AG***
Symmetric: Let xRy so that $x^{-1}y \in H$ ***M1***
it follows that $\left(x^{-1}y\right)^{-1} = y^{-1}x \in H \Rightarrow yRx$ ***M1A1***
therefore symmetric ***AG***
Transitive: Let xRy and yRz so that $x^{-1}y \in H$ and $y^{-1}z \in H$ ***M1***
it follows that $x^{-1}y\, y^{-1}z = x^{-1}z \in H \Rightarrow xRz$ ***M1A1***
therefore transitive (therefore R is an equivalence relation on the set G) ***AG***

[8 marks]

(b) (i) attempt at inverse of 3: since $3 \times 9 = 27 = 1(\text{mod}13)$ ***(M1)***
it follows that $3^{-1} = 9$ ***A1***
since $9 \times 10 = 90 = 12(\text{mod}13) \in H$ ***M1A1***
it follows that $3R10$ ***AG***

(ii) the three equivalence classes are $\{3, 10\}$, $\{1, 12\}$ and $\{4, 9\}$ ***A1A1A1***

[7 marks]

Total [15 marks]

3. (a) (i) the auxiliary equation is $m^2 - m - 6 = 0$ or equivalent ***A1***

(ii) attempt to solve quadratic ***(M1)***
the roots are $3, -2$ ***A1***
the general solution is
$u_n = A \times 3^n + B \times (-2)^n$ ***A1***
initial conditions give
$3A - 2B = 12$
$9A + 4B = 6$ ***M1***
the solution is $A = 2,\ B = -3$ ***A1***
$u_n = 2 \times 3^n - 3 \times (-2)^n$ ***AG***

(iii) $u_n + u_{n-1} = 2 \times 3^n - 3 \times (-2)^n + 2 \times 3^{n-1} - 3 \times (-2)^{n-1}$ ***M1***
$= 8 \times 3^{n-1} +$ multiple of 2^{n-1} ***A1***
$u_n - u_{n-1} = 2 \times 3^n - 3 \times (-2)^n - 2 \times 3^{n-1} + 3 \times (-2)^{n-1}$
$= 4 \times 3^{n-1}$ + multiple of 2^{n-1} ***A1***
any evidence of noting that the 3^{n-1} terms dominate ***R1***
$\lim_{n \to \infty} \frac{u_n + u_{n-1}}{u_n - u_{n-1}} = 2$ ***A1***

[11 marks]

(b) $v_n = 2 \times 3^{2n} - 3 \times (-2)^{2n}$ ***M1***
$= 2 \times 9^n - 3 \times 4^n$ ***A1***
the auxiliary equation is
$m^2 - 13m + 36 = 0$ ***A1***
the recurrence relation is
$v_{n+2} = 13v_{n+1} - 36v_n$ ***A1***

[4 marks]

Total [15 marks]

4. (a) (i) $\det(\boldsymbol{A}) = \lambda(12 - 7\lambda) + 3(3\lambda - 6) + 2(14 - 12)$ ***M1A1***
$= 12\lambda - 7\lambda^2 + 9\lambda - 18 + 4$
$= -7\lambda^2 + 21\lambda - 14$ ***A1***

(ii) $\boldsymbol{A}$ is singular when $\lambda = 1$ because the determinant is zero ***R1***

Note: Do not award the ***R1*** if the determinant has not been obtained.

the other value is 2 ***A1***

[5 marks]

(b) (i) the third row is the sum of the first two rows ***A1***

continued…

Question 4 continued

(ii) the null space satisfies

$$\begin{bmatrix} 1 & 3 & 2 \\ 2 & 4 & 1 \\ 3 & 7 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$ ***M1***

$x + 3y + 2z = 0$

$2x + 4y + z = 0$ ***(A1)***

$3x + 7y + 3z = 0$

the solution is (by GDC or otherwise)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix}\alpha \text{ where } \alpha \in \mathbb{R}$$ ***M1A1***

(iii) the rank-nullity theorem for square matrices states that
rank of matrix + dimension of null space = number of columns ***A1***
here, rank $= 2$, dimension of null space $= 1$ and number of columns $= 3$ ***A1***

[7 marks]

(c) first show that the result is true for $n = 3$

$$\boldsymbol{B}^2 = \begin{bmatrix} 13 & 29 & 11 \\ 13 & 29 & 11 \\ 26 & 58 & 22 \end{bmatrix}$$ ***A1***

$$\boldsymbol{B}^3 = \begin{bmatrix} 104 & 232 & 88 \\ 104 & 232 & 88 \\ 208 & 464 & 176 \end{bmatrix}$$ ***A1***

therefore $\boldsymbol{B}^3 = 8\boldsymbol{B}^2$ so true for $n = 3$ ***R1***

assume the result is true for $n = k$, that is $\boldsymbol{B}^k = 8^{k-2}\boldsymbol{B}^2$ ***M1***

consider $\boldsymbol{B}^{k+1} = 8^{k-2}\boldsymbol{B}^3$ ***M1***

$= 8^{k-2} 8\, \boldsymbol{B}^2$

$= 8^{k-1}\ \boldsymbol{B}^2$ ***A1***

therefore, true for $n = k \Rightarrow$ true for $n = k + 1$ and since the result is true for $n = 3$, it is true for $n \geq 3$ ***R1***

[7 marks]

Total [19 marks]

5. (a) (i) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln(\sec x+\tan x)\right)=\frac{\sec x\tan x+\sec^2 x}{\sec x+\tan x}$ ***M1***

$=\sec x$ ***A1***

therefore $\int \sec x\mathrm{d}x = \ln(\sec x + \tan x) + C$ ***AG***

(ii) $\int \sec^3 x\mathrm{d}x = \int \sec x\times\sec^2 x\mathrm{d}x$ ***M1***

$=\sec x\tan x-\int \sec x\tan^2 x\mathrm{d}x$ ***A1A1***

$=\sec x\tan x-\int \sec x\left(\sec^2 x-1\right)\mathrm{d}x$ ***A1***

$=\sec x\tan x-\int \sec^3 x\mathrm{d}x+\int \sec x\mathrm{d}x$

$=\sec x\tan x-\int \sec^3 x\mathrm{d}x+\ln(\sec x+\tan x)$ ***A1***

$2\int \sec^3 x\mathrm{d}x=\left(\sec x\tan x+\ln(\sec x+\tan x)\right)$ ***A1***

therefore

$\int \sec^3 x\mathrm{d}x=\frac{1}{2}\left(\sec x\tan x+\ln(\sec x+\tan x)\right)+C$ ***AG***

[8 marks]

(b) (i) int factor $=\mathrm{e}^{\int \tan x\mathrm{d}x}$ ***(M1)***

$=\mathrm{e}^{\ln\sec x}$ ***(A1)***

$=\sec x$ ***A1***

the differential equation can be written as

$\frac{\mathrm{d}}{\mathrm{d}x}(y\sec x)=2\sec^3 x$ ***M1A1***

integrating,

$y\sec x = \sec x\tan x + \ln(\sec x + \tan x) + C$ ***A1***

putting $x=0, y=1,$ ***M1***

$C=1$ ***A1***

the solution is $y = \cos x\left(\sec x\tan x + \ln(\sec x + \tan x) + 1\right)$ ***A1***

continued…

Question 5 continued

(ii) differentiating the differential equation,

$$\frac{d^2y}{dx^2}+\frac{dy}{dx}\tan x+y\sec^2 x=4\sec^2 x\tan x$$ ***A1A1***

$$\frac{d^2y}{dx^2}+\left(2\sec^2 x-y\tan x\right)\tan x+y\sec^2 x=4\sec^2 x\tan x$$ ***A1***

$$\frac{d^2y}{dx^2}+y=2\sec^2 x\tan x$$ ***AG***

(iii) at a point of inflection, $\frac{d^2y}{dx^2}=0$ so $y=2\sec^2 x\tan x$ ***(M1)***

therefore the point of inflection can be found as the point of intersection of the graphs of $y=\cos x\left(\sec x\tan x+\ln(\sec x+\tan x)+1\right)$

and $y=2\sec^2 x\tan x$ ***(M1)***

drawing these graphs on the calculator, $x=0.605$ ***A2***

[16 marks]

Total [24 marks]

6. (a) (i) let r = radius of circle. Consider

$\mathrm{PR}\times\mathrm{PS}=(\mathrm{PO}-r)(\mathrm{PO}+r)$ ***M1***

$=\mathrm{PO}^2-\mathrm{OQ}^2$ ***A1***

$=\mathrm{PQ}^2$ because POQ is a right angled triangle ***R1***

(ii) the result is true even if PS does not pass through O ***A1***

[4 marks]

(b) (i) using the tangent-secant theorem, ***M1***

$\mathrm{AD}^2=\mathrm{BD}\times\mathrm{CD}$ ***A1***

so $\frac{\mathrm{AD}^2}{\mathrm{BD}^2}=\frac{\mathrm{CD}}{\mathrm{BD}}\ldots$ (1) ***AG***

continued…

Question 6 continued

(ii) consider the triangles CAD and ABD. They are similar because $\text{D}\hat{\text{A}}\text{B} = \text{A}\hat{\text{C}}\text{D}$, angle $\hat{\text{D}}$ is common therefore the third angles must be equal **_M1_** **_A1_**

Note: Beware of the assumption that AC is a diameter of the circle.

therefore

$\frac{\text{AD}}{\text{BD}} = \frac{\text{AC}}{\text{AB}} \ldots$ (2) **_AG_**

it follows from (1) and (2) that

$\frac{\text{CD}}{\text{BD}} = \frac{\text{AC}^2}{\text{AB}^2}$ **_AG_**

(iii) two similar expressions are

$\frac{\text{AE}}{\text{CE}} = \frac{\text{BA}^2}{\text{BC}^2}$ **_M1A1_**

$\frac{\text{BF}}{\text{AF}} = \frac{\text{CB}^2}{\text{CA}^2}$ **_A1_**

multiplying the three expressions,

$\frac{\text{CD}}{\text{BD}} \times \frac{\text{AE}}{\text{CE}} \times \frac{\text{BF}}{\text{AF}} = \frac{\text{AC}^2}{\text{AB}^2} \times \frac{\text{BA}^2}{\text{BC}^2} \times \frac{\text{CB}^2}{\text{CA}^2}$ **_M1_**

$\frac{\text{CD}}{\text{BD}} \times \frac{\text{AE}}{\text{CE}} \times \frac{\text{BF}}{\text{AF}} = 1$ **_A1_**

it follows from the converse of Menelaus' theorem (ignoring signs) **_R1_**

that D, E, F are collinear **_AG_**

[10 marks]

Total [14 marks]

7. (a) (i) $G(t) = \frac{kt}{3} + \frac{2kt^2}{3^2} + \frac{3kt^3}{3^3} + \ldots$ **_M1A1_**

(ii) $\frac{T_{n+1}}{T_n} = \frac{(n+1)kt^{n+1}}{3^{n+1}} \times \frac{3^n}{nkt^n}$ **_M1A1_**

$\to \frac{t}{3}$ as $n \to \infty$ **_A1_**

for convergence, $\left|\frac{t}{3}\right| < 1$ so radius of convergence $= 3$ **_A1_**

continued…

Question 7 continued

(iii) $G(t) = \frac{kt}{3} + \frac{2kt^2}{3^2} + \frac{3kt^3}{3^3} + \dots$

$\frac{t}{3}G(t) = \frac{kt^2}{3^2} + \frac{2kt^3}{3^3} + \dots$

$\left(1 - \frac{t}{3}\right)G(t) = \frac{kt}{3} + \frac{kt^2}{3^2} + \frac{kt^3}{3^3} + \dots$ ***M1A1***

$= \frac{\frac{kt}{3}}{\left(1 - \frac{t}{3}\right)}$ ***A1***

$G(t) = \frac{\frac{kt}{3}}{\left(1 - \frac{t}{3}\right)^2} = \frac{3kt}{(3-t)^2}$ ***AG***

(iv) $G(1) = 1$ ***M1***

so $k = \frac{4}{3}$ ***AG***

[10 marks]

(b) (i) $\ln G(t) = \ln 4t - \ln(3-t)^2$ ***M1***

$\ln G(t) = \ln 4 + \ln t - 2\ln(3 - t)$ ***AG***

(ii) $\frac{G'(t)}{G(t)} = \frac{1}{t} + \frac{2}{3 - t}$ ***M1A1***

putting $t = 1$

$G'(1) = 2$ ***A1***

$\frac{G''(t)G(t) - \left[G'(t)\right]^2}{\left[G(t)\right]^2} = -\frac{1}{t^2} + \frac{2}{(3-t)^2}$ ***M1A1***

putting $t = 1$

$G''(1) - 4 = -1 + \frac{1}{2}$

$G''(1) = \frac{7}{2}$ ***A1***

(iii) $\mathrm{Var}(X) = G''(1) + G'(1) - \left[G'(1)\right]^2 = \frac{3}{2}$ ***A1***

[8 marks]

Total [18 marks]

8. (a) (i) $a \times_n b = 0 \Rightarrow ab =$ a multiple of n (or vice versa) ***R1***

since n is prime, this can only occur if $a = 1$ and $b =$ multiple of n which is impossible because the multiple of n would not belong to S_n ***R1***

(ii) $a \times_n b = a \times_n c \Rightarrow a \times_n (b - c) = 0$ ***M1***

suppose $b \neq c$ and let $b > c$ (without loss of generality)

$(b-c) \in S_n$ and from (i), $a \times_n (b-c) = 0$ is a contradiction ***R1***

therefore $b = c$ ***AG***

[4 marks]

(b) G_n is associative because modular multiplication is associative ***A1***

G_n is closed because the value of $a \times_n b$ always lies between 1 and $n-1$ ***A1***

the identity is 1 ***A1***

consider $a \times_n b$ where b can take $n-1$ possible values. Using the result from (a)(ii), this will result in $n-1$ different values, one of which will be 1, which will give the inverse of a ***R1***

G_n is therefore a group ***AG***

[4 marks]

(c) (i) $(n-1)^2 = n^2 - 2n + 1 \equiv 1 (\operatorname{mod} n)$ ***M1***

so that $(n-1) \times_n (n-1) = 1$ and $n-1$ has order 2 ***R1AG***

(ii) consider $2 \times \frac{1}{2}(n+1) = n + 1 = 1 (\operatorname{mod} n)$ ***A1***

since $\frac{1}{2}(n+1)$ is an integer for all n, it is the inverse of 2 ***R1AG***

(iii) consider $3 \times \frac{1}{3}(n+1) = n + 1 = 1 (\operatorname{mod} n)$ ***M1***

therefore $\frac{1}{3}(n+1)$ is the inverse of 3 if it is an integer but not otherwise ***R1***

(iv) the inverse of 3 in G_{11} is 4 ***A1***

(v) the inverse of 3 in G_{31} is 21 ***(M1)A1***

[9 marks]

Total [17 marks]

9. (a) consider $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix}$ ***(M1)***

the image of $(1, 0)$ is $(\cos\alpha, \sin\alpha)$ ***A1***

therefore $a = \cos\alpha$, $c = \sin\alpha$ ***AG***

consider $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}$

the image of $(0, 1)$ is $(-\sin\alpha, \cos\alpha)$ ***A1***

therefore $b = -\sin\alpha$, $d = \cos\alpha$ ***AG***

[3 marks]

(b) (i) $\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix}$

or $x = X\cos\alpha + Y\sin\alpha$, $y = -X\sin\alpha + Y\cos\alpha$ ***A1***

substituting in the equation of the hyperbola, ***M1***

$(X\cos\alpha + Y\sin\alpha)^2 - 4(X\cos\alpha + Y\sin\alpha)(-X\sin\alpha + Y\cos\alpha)$

$-2(-X\sin\alpha + Y\cos\alpha)^2 = 3$ ***A1***

$X^2\left(\cos^2\alpha - 2\sin^2\alpha + 4\sin\alpha\cos\alpha\right) +$

$XY\left(2\sin\alpha\cos\alpha - 4\cos^2\alpha + 4\sin^2\alpha + 4\sin\alpha\cos\alpha\right) +$

$Y^2\left(\sin^2\alpha - 2\cos^2\alpha - 4\sin\alpha\cos\alpha\right) = 3$

(ii) when $\tan\alpha = \frac{1}{2}$, $\sin\alpha = \frac{1}{\sqrt{5}}$ and $\cos\alpha = \frac{2}{\sqrt{5}}$ ***A1***

the XY term $= 6\sin\alpha\cos\alpha - 4\cos^2\alpha + 4\sin^2\alpha$ ***M1***

$= 6 \times \frac{1}{\sqrt{5}} \times \frac{2}{\sqrt{5}} - 4 \times \frac{4}{5} + 4 \times \frac{1}{5}\left(\frac{12}{5} - \frac{16}{5} + \frac{4}{5}\right)$ ***A1***

$= 0$ ***AG***

(iii) the equation of the rotated hyperbola is

$2X^2 - 3Y^2 = 3$ ***M1A1***

$\frac{X^2}{\left(\sqrt{\frac{3}{2}}\right)^2} - \frac{Y^2}{(1)^2} = 1$ ***A1***

(Accept $\frac{X^2}{\frac{3}{2}} - \frac{Y^2}{1} = 1$)

continued…

Question 9 continued

(iv) the coordinates of the foci of the rotated hyperbola

are $\left(\pm\sqrt{\frac{3}{2}+1},\ 0\right)=\left(\pm\sqrt{\frac{5}{2}},\ 0\right)$ ***M1A1***

the coordinates of the foci prior to rotation were given by

$$\begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}\begin{bmatrix} \pm\sqrt{\frac{5}{2}} \\ 0 \end{bmatrix}$$ ***M1A1***

$$\begin{bmatrix} \pm\sqrt{2} \\ \mp\frac{1}{\sqrt{2}} \end{bmatrix}$$ ***A1***

[14 marks]

Total [17 marks]